Math 821, Spring 2013

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What about the antipode.

Proposition.
(a) $S(p_n) = -p_n$
(b) $S(e_n) = (-1)^n h_n$
(c) $S(h_n) = (-1)^n e_n$
Recall the fundamental involution $\omega : \bigwedge \to \bigwedge : e_n \mapsto h_n$, $\omega \circ \omega = id$.

Proof of Proposition.
(a) From the fact that p_n are primitive.
(b, c) Recall from last time we had $\Delta e_n = \sum_{i+j=n} e_i \otimes e_j$ and $\Delta h_n = \sum_{i+j=n} h_i \otimes h_j$ so we also have $S * id = u \circ \epsilon = id * S$ applied to e_n. $\sum_{i+j=n} S(e_i)e_j = \delta_{0,n} = \sum_{i+j=n} e_i S(e_j)$ where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

is the dirac delta function. Likewise $\sum_{i+j=n} S(h_i)h_j = \delta_{n,0} = \sum_{i+j=n} h_i S(h_j)$. From the fundamental involution $\sum_{i+j=n} (-1)^i e_i h_j = \delta_{0,n}$ so by induction or independence $S(h_i) = (-1)^i e_i$. Then since ω is an involution, we also get $S(e_i) = (-1)^i h_i$.

cor $S(f) = (-1)^n \omega(f)\ \forall f \in \bigwedge_n$.

Robinson-Schensted Algorithm

Definition. A (standard) Young tableau of shape λ, λ a partition of n is a filling of the Ferrer's diagram of λ with $\{1, 2, \ldots, n\}$ and strictly increasing along rows and columns.
ex $\lambda = (4, 2, 1)$, $n = 7$.

1	3	4	7
2	5		
6			

Theorem.
There is a bijection between permutations of $\{1, \ldots, n\}$ and pairs of Young tableaux of the same shape λ where λ is a partition of n. The bijection is given by an algorithm.

Definition.
Given a tableau T with distinct entries and i not in T, then the procedure to insert i into T is as follows. To begin the current row is the top row of T.
· find the smallest $j > i$ in current row if it exists
· if no such j exists put a new box at the end of the current row and put i in it. STOP.
· otherwise put i where j was and continue with the next row down as the current row and j as i.

ex

$$
\begin{array}{ll}
T & \text{i} \\[6pt]
\begin{array}{|c|c|c|}\hline 2 & 5 & 6 \\ \hline 7 & 8 \\ \cline{1-2} \end{array} & \leftarrow 3 \\[6pt]
\begin{array}{|c|c|c|}\hline 2 & 3 & 6 \\ \hline 7 & 8 \\ \cline{1-2} \end{array} & \leftarrow 5 \\[6pt]
\begin{array}{|c|c|c|}\hline 2 & 3 & 6 \\ \hline 5 & 8 \\ \cline{1-2} \end{array} & \leftarrow 7 \\[6pt]
\begin{array}{|c|c|c|}\hline 2 & 3 & 6 \\ \hline 5 & 8 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} &
\end{array}
$$

is the result of the procedure. Observe that the procedure terminates because each time through we go down a row, but there are only finitely many nonempty rows. So if I don't stop sooner in a finite length of time, I'll reach an empty row at which point I will necessarily arrive at bullet 2 and so terminate.

Robinson Schensted Algorithm.
Input σ a permutation of $\{1, \ldots, n\}$
Set P_0, Q_0 empty tableaux
for i from 1 to n
· insert $\sigma(i)$ into P_{i-1}, call the result P_i
· P_i differs in shape from P_{i-1} in exactly one added box. Take Q_{i-1} add a box in that position, put i in it and call the result Q_i.
Output P_n and Q_n.

ex
Write the permutation in 2−line notation

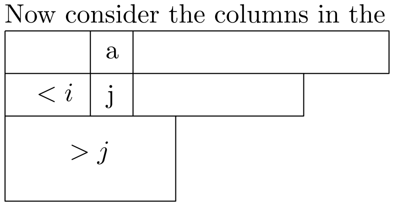
$$
L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 7 & 2 & 8 & 6 & 1 & 4 & 3 & 9 \end{pmatrix}
$$

i	$\sigma(i)$	P_i	Q_i
1	5	5	1
2	7	5 7	1 2
3	2	2 7 5	1 2 3
4	8	2 7 8 5	1 2 4 3
5	6	2 6 8 5 7	1 2 4 3 5
6	1	1 6 8 2 7 5	1 2 4 3 5 6
7	4	1 4 8 2 6 5 7	1 2 4 3 5 6 7
8	3	1 3 8 2 4 5 6 7	1 2 4 3 5 6 7 8
9	9	1 3 8 9 2 4 5 6 7	1 2 4 9 3 5 6 7 8

Observe P_i and Q_i are always the same shape by induction. Also each time through the for loop adds a box to P_i and Q_i so P_n, Q_n each contain n boxes. Next we want to show that insertion maintains the tableau property.

In the bump case $j > i$ but the smallest such j in the row so by distinctness i is larger than everything preceeding j in the row and by choice of j is smaller than everything after.

Now consider the columns in the bump case.



Any element below j is $> j$ and hence $> i$ so no problem. If we are in first row, no problem. Otherwise i came from bumping in the row above. i could not have been in a column to the left of j because in j's row all elements to the left of j are $< i$ by the above paragraph. So either i was above j or something $< i$ was above j. When it is bumped something smaller goes in its place so after that the element above j is $< i$.

Next consider the case when a box is added to a row. In this case the nonstrictness is maintained as no element in the row is $> j$ and elements are distinct. Consider the columns, if we're in the top row then done. Otherwise we need to make sure that there is a box above the new box and it contains something $< i$. But to have gotten to this point i must have been bumped from the row above but i can't have been above anything in the current row since those elements are $< i$. So either i was above the new box or something

$< i$ was. But i was bumped by something less than itself so after bumping something $< i$ is above the new box. So the tableau property is maintained so by induction each P_i is a tableau. The Q_i are also tableau since at each stage I add a new box at the end of a row or begin a new row and put a larger number in. It remains to show this is a bijection. We can do so by showing it's reversible.

<u>ex</u>

$$P \qquad \begin{array}{|c|c|c|c|}\hline 1&3&8&9\\\hline 2&4\\\cline{1-2} 5&6\\\cline{1-2} 7\\\cline{1-1}\end{array} \qquad\qquad Q \qquad \begin{array}{|c|c|c|c|}\hline 1&2&4&9\\\hline 3&5\\\cline{1-2} 6&7\\\cline{1-2} 8\\\cline{1-1}\end{array}$$

Here's how to reverse a step give P_i, Q_i.

Find the largest element k in P_i. Find the matching box in Q_i, call the entry in the box l. Remove l from Q_i to get Q_{i-1}. Uninsert l from P_i to get P_{i-1}. The entry which pops out is l' then $k \mapsto l'$ in the permutation. Continuing the example

$$\begin{array}{ccc} & P_i & Q_i \\ 7 \to 6 & \begin{array}{|c|c|c|}\hline 1&3&8\\\hline 2&4\\\cline{1-2} 5&6\\\cline{1-2} 7\\\cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6&7\\\cline{1-2} 8\\\cline{1-1}\end{array} \\ 6 \to 4 & \begin{array}{|c|c|c|}\hline 1&3&8\\\hline 2&4\\\cline{1-2} 5&7\\\cline{1-2}\end{array} & \\ 4 \to 3 & \begin{array}{|c|c|c|}\hline 1&3&8\\\hline 2&6\\\cline{1-2} 5&7\\\cline{1-2}\end{array} & \\ 3 \text{ leaves} & \begin{array}{|c|c|c|}\hline 1&4&8\\\hline 2&6\\\cline{1-2} 5&7\\\cline{1-2}\end{array} & \end{array}$$

How to uninsert j

· remove j and its box

· in the row above find the largest entry i, $i < j$ (always exist if there is a row) replace i by j.

· now do the same in the row above with i in place of j

When we process the top row the entry we find is the l' which pops out.

The inverse $\quad \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 8 & 3 & 7 & 1 & 5 & 2 & 4 & 9 \end{matrix}$

$$\begin{array}{|c|c|c|c|}\hline 1&2&4&9\\\hline 3&5\\\cline{1-2} 6&7\\\cline{1-2} 8\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&3&8&9\\\hline 2&4\\\cline{1-2} 5&6\\\cline{1-2} 7\\\cline{1-1}\end{array}$$

Note P and Q are switched. This is true in general but requires another construction.

One way to view a permutation is as a permutation matrix.

<u>ex</u>

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Permutation matrices are matrices with exactly one 1 in each row and exactly one 1 in each column and rest are 0's. Instead consider matrices with all nonnegative integer entries.

ex $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Convert to a list of pairs as before: if the (i,j)th entry is k, put k copies of $\begin{bmatrix} i \\ j \end{bmatrix}$ in the list ordered lexicographically (i.e. ordered first by top entry and among pairs with the same top entry, ordered by bottom entries). This example yields

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 & 3 \end{bmatrix}$$

Next time we'll prove

Theorem.
There is a bijection between lists of ordered pairs of positive integers ordered lexicographically, length of list is n and pairs of semistandard Young tableaux of the same shape λ where λ is a partition of n.

References. (Sketch in Reiner 2.5).